Math 131 B, Lecture 2
Analysis

Midterm 2

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

(a) [5pts.] Let (M, d_M) and (S, d_S) be two metric spaces. Give a definition of a continuous function $f : M \to S$.

(b) [5pts.] For each of the following pairs of metric spaces, either construct a continuous function $f : M \to S$ with $f(M) = S$ or explain why one cannot exist.

- (a) $M = (3, 5)$, $S = \mathbb{Q}$.
- (b)$M = [0, 1]$, $S = C([0, 1] \to \mathbb{R})$.
- (c)$M = \mathbb{R}$, $S = \{(x, y) : x^2 + y^2 = 1\}$.
- (d)$M = (1, 2) \cup (3, 4)$, $S = \{0, 1, 2\}$.

Problem 2.

(a) [5pts.] Give a definition of a connected metric space M.

(b) [4pts.] Prove that the intersection of two connected subsets of the real line is connected.

(c) [1pts.] Give an example showing that the above result need not be true in an arbitrary metric space. (A sketch is fine, you don't need to prove it.)

Problem 3.

(a) [5pts.] Let $\{f_n\}$ be a sequence of functions $f_n : S \to T$. What does it mean for f_n to converge uniformly to a function $f : S \to T$?

(b) [5pts.] Prove that if $f_n \to f$ uniformly and each f_n is continuous, f is continuous.

Problem 4.

(a) [5pts.] State Abel's Theorem.

(b) [5pts.] Prove that the sum

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

converges to $\frac{\pi}{4}$.

Problem 5.

(a) [5pts.] State the Weierstrass M-test.

(b) [5pts.] Prove that the series $\sum_{n=2}^{\infty} \ln(1 + \frac{x}{n^2})$ converges uniformly on $(-1, 1)$. [Hint: How do the derivatives behave?]